Dynamic Epistemic Logic and Category Theory

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Abstract

Epistemic models and action epistemic models are two important structures in epistemic logic. We present new categories with epistemic models and action epistemic models. We name these categories epistemic categories. With these categories we try to study epistemic logic in an abstract way. For example we try to answer the following question that how we can combine the concept of time with epistemic logic. Positive answer to this question helps us to study protocols. Also in a similar and more abstract way we present new subcategories in the category of measurable spaces. These categories can capture epistemic changes. We use syntax, semantics and deduction systems of coalgebras on these categories to help us in our studies. Indeed we present a logical core for different epistemic changes.

Keywords: Epistemic logic, Dynamic epistemic logic, Category theory, Epistemic category, Measurable space, Coalgebra.

1 Introduction

Epistemic logic was first introduced by Hintikka. Hintikka presented a logic for knowledge and belief [11]. Epistemic logic is a logic for reasoning in knowledge. For better studies of knowledge, epistemic logic was extended to dynamic epistemic logic [2,1]. Dynamic epistemic logic has action operators for capturing epistemic actions. Dynamic epistemic logic focuses on epistemic changes. For example with dynamic epistemic logic we can capture belief changes and belief revision theory in dynamic version [5,6]. There have been efforts for combining dynamic epistemic logic and time. For example Van Benthem has studied the merging of dynamic epistemic logic and temporal epistemic logic [7]. By using category theory and coalgebras we extend these epistemic studies. First, we introduce categories with epistemic models as objects and action epistemic models as arrows. We study categorical properties of these categories. Also we show how an epistemic model can define a measurable space and how an action epistemic model can introduce a measurable function. We can introduce a subcategory in measurable spaces in similar way. Indeed we can introduce measurable spaces and measurable functions by Kripke models and action Kripke

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models, and define a category. Moss and Viglizzo studied coalgebras on measurable spaces. They introduced a semantics and syntax for coalgebras on measurable spaces as coalgebras on the category of sets [16,19,17]. Also Goldblatt presented deduction systems for these coalgebras on measurable spaces [10]. We will use these studies for our purpose. We will use functors for formalizing the concept of time. For example, it has been shown that how linear time can be captured in fan categories which are models for intiounistic logic. We can capture different (polynomial) functors which can capture different changes. So we can introduce a logical core for different changes particularly epistemic changes.

2 Definitions

In this section we introduce some important definitions in epistemic logic [8,6].

Definition 2.1 [Single-agent epistemic model] A single-agent epistemic frame is a structure $(S, \sim)$ consisting of a set S of "states" and an "equivalence relation" $\sim$, i.e., a reflexive, transitive and symmetric binary relation on S. We can define an epistemic model to be a structure $\mathbf{S} = (S, \sim, \| . \|)$ consisting of an epistemic frame $(S, \sim)$ with a valuation map (on atomic sentences) $\| . \|: \Phi \longrightarrow \mathcal{P}(S)$.

Definition 2.2 [S-proposition] Given an epistemic frame S, an S-proposition is any subset $P \subseteq S$. We say that a state s satisfies the proposition P if $s \in P$.

Definition 2.3 [Doxastic proposition] A doxastic proposition is a map $\mathbf{P}$ assigning to each epistemic model $\mathbf{S}$ some S-proposition $\mathbf{P_S}$. We write $s \models_{\mathbf{S}} \mathbf{P}$ iff $s \in (\mathbf{P})_{\mathbf{S}}$, and we say that the proposition $\mathbf{P}$ is true at $s \in \mathbf{S}$.

Definition 2.4 [Single-agent action epistemic model] A single-agent action epistemic model is an epistemic frame $(\Sigma, \sim)$ together with a precondition map $pre : \Sigma \longrightarrow Prop$, associating to each element of Σ some doxastic proposition pre_σ.

Definition 2.5 [Product update (for single-agent)] Let $(S, \sim, \| . \|)$ be a single-agent epistemic model and let $(\Sigma, \sim', pre)$ be a single-agent action epistemic model. The set of states of updated model $\mathbf{S} \bigotimes \mathbf{\Sigma}$ is taken to be:

$$\mathbf{S} \bigotimes \mathbf{\Sigma} := \{ (s, \sigma): s \models_{\mathbf{S}} pre(\sigma) \}.$$

For $(s, \sigma) \in \mathbf{S} \bigotimes \mathbf{\Sigma}$ put: $(s, \sigma) \models p$ iff $s \models p$.

Definition 2.6 [Isomorphic models (for single-agent)] We say two epistemic models $\mathbf{S} = (S, \sim, \| . \|)$ and $\mathbf{S}' = (S', \sim', \| . \|')$ are isomorphic if there is a bijective function from S to S' $(f : \mathbf{S} \longrightarrow \mathbf{S}')$ which satisfies the following conditions:
1) For any atomic sentence p and possible word s : $p \in \| s \|$ iff $p \in \| f(s) \|'$;
2)$s \sim s'$ iff $f(s) \sim' f(s')$.

Definition 2.7 [Information cell (for single-agent)] Every epistemic relation $\sim$ induces a partition of the state space S. We denote the information cell for agent a and possible state s by $s(a)$.

$$s(a) := \{t \in S : s \sim t\}.$$

3 Epistemic Category

We introduce an epistemic category C, as follow:

Objects. All finite single-agent epistemic models (up to isomorphism) with one information cell such that:
$\mathbf{\Omega}$: For s in $\mathbf{S} = (S, \sim, \| . \|)$ and s' in $\mathbf{S}' = (S', \sim', \| . \|')$ if we have: $\forall p$ $(s \in \| p \|$ iff $s' \in \| p \|')$, for any doxastic proposition we will have: $s \models_{\mathbf{S}} \mathbf{P}$ iff $s' \models_{\mathbf{S}'} \mathbf{P}$.

Remark. This condition is a skepticism condition. Two states that have different theory (I mean that one of them satisfies at least a different doxastic proposition.) must have a different experimental subject (I mean that one of them satisfies at least a different atomic proposition (ontic fact).).

Arrows. All single-agent action epistemic models with one member.

Well-definition. To see that this category is well-defined we show if $\mathbf{S} \cong \mathbf{S}'$, for any action epistemic model $\mathbf{\Sigma} = (\{\sigma\}, \sim, pre)$ we have $\mathbf{S} \bigotimes \mathbf{\Sigma} \cong \mathbf{S}' \bigotimes \mathbf{\Sigma}$. For $\mathbf{S} \cong \mathbf{S}'$ there is an isomorphism $f : \mathbf{S} \longrightarrow \mathbf{S}'$. So we can consider an isomorphism between $\mathbf{S} \bigotimes \mathbf{\Sigma}$ and $\mathbf{S}' \bigotimes \mathbf{\Sigma}$ such as:

$$(s, \sigma) \xrightarrow{g} (f(s), \sigma).$$

By considering the $\mathbf{\Omega}$ condition, s and $f(s)$ satisfy the same doxastic propositions. So g define an isomorphism between two models.
We will show that C is a category.

Identity arrow. For any epistemic model $\mathbf{S} = (S, \sim, \| . \|)$, we define action epistemic model $\bigotimes \mathbf{\Sigma}$ with single-member frame $(\sigma_*, \sim')$ as below:

$pre(\sigma_*) = \mathbf{P}^*$;
$\mathbf{P}^*{}_{\mathbf{S}} = S$;
$\bigotimes \mathbf{\Sigma}(\mathbf{S}) := \mathbf{S} \bigotimes \mathbf{\Sigma} := \{(s, \sigma) : s \models_{\mathbf{S}} pre(\sigma)\} := \{(s, S) : s \models_{\mathbf{S}} \mathbf{P}^*{}_{\mathbf{S}}\}$
$:= \{(s, S) : s \in S\} \cong \mathbf{S}$.

with the definition of updated model the new model relation $\sim''$ will be defined as below:

$$(s, \sigma_*) \sim'' (s', \sigma_*) \quad \text{iff} \quad \sigma_* \sim' \sigma_* \text{ and } s \sim s'.$$

Composition. The composition of $\bigotimes \mathbf{\Sigma}_1$ and $\bigotimes \mathbf{\Sigma}_2$ in according to the theorem "The updated model outcome of act an action epistemic and an epistemic model is an epistemic model" exists as below:

$$\bigotimes \mathbf{\Sigma}_1 \bigotimes \mathbf{\Sigma}_2 = \bigotimes \mathbf{\Sigma}_1 (\bigotimes \mathbf{\Sigma}_2).$$

Associativity. By considering the definition of product updat, we see:

$$\bigotimes \mathbf{\Sigma}_1 \,(\, \bigotimes \mathbf{\Sigma}_2 \bigotimes \mathbf{\Sigma}_3 \,) = \,(\, \bigotimes \mathbf{\Sigma}_1 \bigotimes \mathbf{\Sigma}_2 \,) \bigotimes \mathbf{\Sigma}_3.$$

Since we have:

$$\{\, (\,(\,(s, \sigma_1\,), \sigma_2\,), \sigma_3\,) \colon s \models pre\sigma_3,\ (s, \sigma_3\,) \models pre\sigma_2,\ (\,(s, \sigma_3\,), \sigma_2\,) \models pre\sigma_1 \,\},$$

for every s in epistemic model $\mathbf{S}$.

Unit. For every epistemic model $\mathbf{S}$ and an action epistemic model $\bigotimes \mathbf{\Sigma}$ we have:

$$\bigotimes \mathbf{\Sigma} \bigotimes \mathbf{1_S} = \bigotimes \mathbf{1_S} \bigotimes \mathbf{\Sigma} = \bigotimes \mathbf{\Sigma}.$$

So C is a category.

4 Epistemic Category $C_{\|.\|_{\mathcal{S}}}$

Now we want to restrict our attention to epistemic models and action epistemic models. Consider the set $\mathcal{S}$ of countable states that has the valuation $\|\,.\,\|_{\mathcal{S}}$ (On countable set of atomic proposition) such that for any two states s_1, s_2 we will have $\exists p \quad s_1 \in \|\, p \,\|_{\mathcal{S}}$ but $s_2 \notin \|\, p \,\|_{\mathcal{S}}$. We define the category $C_{\|.\|_{\mathcal{S}}}$ as follows:

Objects. All finite single-agent epistemic models $(S, \sim, \|\,.\,\|)$ (up to isomorphism) with one information cell such that $S \subseteq \mathcal{S}$ and $\|\,.\,\|$ is the restriction of $\|\,.\,\|_{\mathcal{S}}$ on S and models satisfy Ω.

Arrows. All single-agent action epistemic models with one member such that $\Sigma \subseteq \mathcal{S}$.

It is provable that $C_{\|.\|_{\mathcal{S}}}$ is a category similar C.

Definition 4.1 [$\boldsymbol{\mathcal{S}}$] If in the epiastemic model $(\mathcal{S}, \sim, \|\,.\,\|_{\mathcal{S}})$ every two states s and t be equivalent we will denote $(\mathcal{S}, \sim, \|\,.\,\|_{\mathcal{S}})$ by $\boldsymbol{\mathcal{S}}$.

4.1 Categorical properties

Definition 4.2 We say two arrows $\bigotimes \mathbf{\Sigma}_1$ and $\bigotimes \mathbf{\Sigma}_2$ are equal if and only if we have $\boldsymbol{\mathcal{S}} \bigotimes \mathbf{\Sigma}_1 \cong \boldsymbol{\mathcal{S}} \bigotimes \mathbf{\Sigma}_2$.

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Remark. Indeed this condition is a way for talking about equivalent actions. We say two actions is equivalent if and only if they have same effect on initial mode. Intuitively we can say two actions is equivalent in universe if and only if they have same effect from Big Bang.

Well-definition. We have to show if two arrows $\bigotimes \mathbf{\Sigma}_1$ and $\bigotimes \mathbf{\Sigma}_2$ are equal, for an epistemic model $\mathbf{S} = (S, \sim, \|\,.\,\|)$ we have $\mathbf{S} \bigotimes \mathbf{\Sigma}_1 \cong \mathbf{S} \bigotimes \mathbf{\Sigma}_2$. From $\boldsymbol{\mathcal{S}} \bigotimes \mathbf{\Sigma}_1 \cong \boldsymbol{\mathcal{S}} \bigotimes \mathbf{\Sigma}_2$ we can conclude that there is an isomorphism between these two models. Now we consider the restriction of f on S, with respect to this restriction and $\mathbf{\Omega}$ we can define an isomorphism between $\mathbf{S} \bigotimes \mathbf{\Sigma}_1$ and $\mathbf{S} \bigotimes \mathbf{\Sigma}_2$.

Theorem 4.3 *$\boldsymbol{\mathcal{S}}$ is the initial object of* $C_{\|.\|_S}$

Proof. We show that for every model $\mathbf{S}$ there is a unique arrow $f : \boldsymbol{\mathcal{S}} \longrightarrow \mathbf{S}$. We can construct the unique arrow as below:

$$\Sigma = (\left\{\sigma_*\right\}, \sim');$$
$$pre\,(\sigma_*) = \mathbf{P}\,,\ \mathbf{P}_{\mathbf{S}'} = S \cap S'.$$

With the definition of updated model and as models have one information cell the new model relation $\sim''$ will be defined as below:

$$(s, \sigma_*) \sim'' (s', \sigma_*) \text{ iff } \quad \sigma_* \sim' \sigma_* \text{ and } s \sim s'.$$

This model is isomorphic with S by $s \xrightarrow{f} (s, \sigma_*)$. The uniqueness of this arrow is obvious by definition. □

Theorem 4.4 (Colimit for $C_{\|.\|_S}$) *The category* $C_{\|.\|_S}$ *has colimits.*

Proof. Consider cocons $\{f_i : d_i \longrightarrow \mathbf{S}\}$ on diagram D. The cocone $\{f_i : d_i \longrightarrow \boldsymbol{\mathcal{S}}\}$ is a colimit for diagram D since for the cocone $\left\{f'_i : d_i \longrightarrow \mathbf{S}'\right\}$ there is exactly one arrow $f : \boldsymbol{\mathcal{S}} \longrightarrow \mathbf{S}'$ ($\boldsymbol{\mathcal{S}}$ is initial object). Also we have:

$$\forall d_i \in D f \circ f_i = f'_i.$$

The cocone $\{f_i : d_i \longrightarrow \boldsymbol{\mathcal{S}}\}$ has only one identity arrow $id : \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$. □

So the category $C^{op}{}_{\|.\|_S}$ is complete. (We showed that $C_{\|.\|_S}$ has an initial object and $C^{op}{}_{\|.\|_S}$ is a complete category. Also we have "a small category C is complete if and only if it is cocomplete" [15]. So $C_{\|.\|_S}$ is a complete category and has products and coproducts.)

Remark. We can consider the identity arrow ($id : \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$) as stability property of a system. In this sense stability of a system can be an important role in other properties of a system.

4.2 Multi-agent epistemic models with several information cells

We have studied epistemic categories by single-agent models with one information cell and single-agent action models with one member. We can extend our studies to multi-agent models with several information cells and action models with several members.

Remark. For a multi-agent epistemic model $(S, \sim_a, \|\,.\,\|)$ and an action epistemic model $(\Sigma, \sim_a, pre)$ we can introduce relation $\sim_a$ with index for agent a. In these models we can define a cell information for agent a as $s(a) := \{t \in S : s \sim_a t\}$. Also product update in multi-agent model can be introduced for any relation $\sim_a$. When an action epistemic model has several members and two actions σ_1 and σ_2 can act on state s, we denote the correspondent state of these two actions by $(s, \sigma_1 \vee \sigma_2)$, since in bases of $\mathbf{\Omega}$ two states (s, σ_1) and (s, σ_2) are equal. Also for the update product rule, $(s, \bigvee_{i\in K}\sigma_i) \sim (s', \bigvee_{j\in L}\sigma_j)$ iff $s \sim s'$ and $\bigvee_{i\in K}\sigma_i \sim \bigvee_{j\in L}\sigma_j$, and $(\bigvee_{i\in K}\sigma_i \sim \bigvee_{j\in L}\sigma_j)$ iff $\exists i \in K, \exists j \in L \quad \sigma_i \sim \sigma_j$.

In different conditions we can introduce other epistemic categories that have same categorical properties as $C_{\|.\|_{\mathcal{S}}}$. In below categories for objects we have that states of epistemic models belong to a fixed set $\mathcal{S}$ and models satisfy Ω. These categories have finite products, coproducts, initial and terminal objects. Checking these properties is similar to checking properties of $C_{\|.\|_{\mathcal{S}}}$.

- $C^1_{\|.\|_{\mathcal{S}}}$; **Objects:** All finite multi-agent epistemic models with one information cell, **Arrows:** All multi-agent action epistemic models with one member.
- ${C^2}_{\|.\|_{\mathcal{S}}}$; **Objects:** All finite multi-agent epistemic models with one information cell, **Arrows:** All multi-agent action epistemic models.
- ${C^3}_{\|.\|_{\mathcal{S}}}$; **Objects:** All finite single-agent epistemic models, **Arrows:** All single-agent action epistemic models.
- ${C^4}_{\|.\|_{\mathcal{S}}}$; **Objects:** All finite multi-agent epistemic models, **Arrows:** All multi-agent action epistemic models.

Remark. In epistemic categories that models (objects) have several information cells the arrow from initial object to other objects can defined as below:
For epistemic model $\mathbf{S} = (S, \sim_a, \| . \|)$ we consider action epistemic model$\boldsymbol{\Delta}$ such that its frame$(\Sigma, \sim_a)$ is isomorphic to $(S, \sim_a)$ with $s_i \longrightarrow \sigma_i$ Also for *pre* we have:

$$pre\,(\sigma_i) = \mathbf{P}_i,\ (\mathbf{P}_i)_{\mathbf{S}} = \{s_i\}.$$

5 Modeling Temporal Modalities in Categorical Context

For modeling temporal modalities $\Box$ and $\Diamond$ we consider a finite set T with a total order $\leq$. We can define $<$ and $>$ naturally. $t < t^{'}$ intuitively means that at time t, $t^{'}$ occurs in the future. A formula $\Box\varphi$ Means that φ holds now and at every future time (finitely) and a formula $\Diamond\varphi$ means that φ holds now or at some future time (finitely). So a proposition $(\Box\varphi)(t)$ corresponds to (finitely) conjunction of all $\varphi(t^{'})$ with $t^{'} > t$, while a proposition $(\Diamond\varphi)(t)$ corresponds to disjunction of all $\varphi(t^{'})$ with $t^{'} > t$. So we can define modal operators $\Box$ and $\Diamond$ as a function as below (see [13]):

$$(\Diamond A)(t) = \coprod_{t\leqslant t'} A(t^{'}),\ (\Box A)(t) = \prod_{t\leqslant t'} A(t^{'}).$$

For a category that has finite products and coporducts we can define temporal functors as below:

$$(\Diamond f)(t) = \coprod_{t\leqslant t'} f(t^{'}),\ (\Box f)(t) = \prod_{t\leqslant t'} f(t^{'}).$$

Also we can generalize our operators ("always" and" eventually" operators in temporal logic) with "until" operator (see [14]):
$\varphi \rhd'' \psi$ holds if ψ will hold at some future time, and φ will hold until ψ holds.
$\varphi \blacktriangleright'' \psi$ holds if φ will hold forever, in which case ψ is not required to hold at any future time.

We can model the logical operators $\rhd''$ and $\blacktriangleright''$ by two functors as below:
$(f \rhd'' g)_t = \coprod_{t' \in (t,\infty)}((\prod_{t'' \in (t,t')} f_{t''}) \times g_{t'})$
$(f \blacktriangleright'' g)_t = (f \rhd'' g)_t + \prod_{t' \in (t,\infty)} f_{t'}$
Also we can define propositions $\varphi \rhd' \psi$ and $\varphi \blacktriangleright' \psi$ similar to $\varphi \rhd'' \psi$ and $\varphi \blacktriangleright'' \psi$, but require φ to also hold at the present time. Propositions $\varphi \rhd \psi$ and $\varphi \blacktriangleright \psi$ additionally hold if ψ holds at the present time, in which case φ is not required to hold at any time.

$$\varphi \rhd' \psi = \varphi \wedge \varphi \rhd'' \psi \qquad \varphi \blacktriangleright' \psi = \varphi \wedge \varphi \blacktriangleright'' \psi$$
$$\varphi \rhd \psi = \psi \vee \varphi \rhd' \psi \qquad \varphi \blacktriangleright \psi = \psi \vee \varphi \blacktriangleright' \psi$$

A proposition $\Box' \varphi$ holds if φ will always hold, and a proposition $\Diamond' \varphi$ holds if φ will hold at some future time. A proposition $\Box\varphi$ requires φ to also hold at the present time, and a proposition $\Diamond\varphi$ additionally holds if φ holds at the present time.

$$\Box' \varphi = \varphi \blacktriangleright'' \bot \qquad \Diamond' \varphi = \top \rhd' \varphi$$
$$\Box \varphi = \varphi \blacktriangleright' \bot \qquad \Diamond \varphi = \top \rhd \varphi$$

Also derived temporal functors are as below:

$$f \rhd' g = f \times f \rhd'' g \qquad f \blacktriangleright' g = f \times f \blacktriangleright'' g$$
$$f \rhd g = g + f \rhd' g \qquad f \blacktriangleright g = g + f \blacktriangleright' g$$
$$\Box' f = f \blacktriangleright'' 0 \qquad \Diamond' f = 1 \rhd' f$$
$$\Box f = f \blacktriangleright' 0 \qquad \Diamond f = 1 \rhd f$$

These functors are used for modeling linear-time temporal logic on different categories [13,14]. We can apply (for finite T) these functors on epistemic categories $C_{\|.\|_S}$ ($C_{\|.\|_S}$ has finite products and coproducts). It seems that category is an appropriate context for combining the linear concept of time with epistemic logic. We will use these temporal functors in our studies.

6 Measure Theoretic Approach to Epistemic Models and Action Epistemic Models

In this section we want to introduce a subcategory in measurable space category by epistemic models and action epistemic models.

Definition 6.1 [Measurable space] Let $\mathcal{A}$ be a (Boolean) algebra on a set X, i.e. a non-empty collection of subsets of X closed under complements and binary unions. $\mathcal{A}$ is a σ-algebra if it is also closed under countable unions. Then $X = (X, \mathcal{A})$ is called a measurable space and the members of $\mathcal{A}$ are its measurable sets.

Definition 6.2 [Measurable function] A measurable function $f : (X, \mathcal{A}) \longrightarrow (X\prime, \mathcal{A}^{'})$ is a function $f : X \longrightarrow X^{'}$ that the inverse of every measurable set is a measurable set. For this it suffices that $f^{-1}(A) \in \mathcal{A}$ for all sets A in some generating subset of $\mathcal{A}^{'}$.

Theorem 6.3 *Every epistemic model can produce a measurable space and every action epistemic model can produce a measurable function.*

Proof. Every epistemic model is a frame with a valuation $\mathcal{V} : \Phi \longrightarrow P(M)$ such that we can consider the valution as a non-empty collection of subsets of M like below:

$$\mathcal{M} = (M, R, \{X_p\}_{p \in Prop} = \mathcal{A}).$$

$\mathcal{A}$ is a σ algabra (we consider that M is finite) and $\mathcal{A}$ is a non-empty collection of subsets of M that is closed under complements and binary unions.

$$X_p \cup X_q = X_{p \cup q};$$
$$X^c{}_p = X_{\neg p}.$$

So every finite epistemic model (without its relation) can define a measurable space. Also we can consider different measures on M, $\mu : \mathcal{A} \longrightarrow [0, \infty]$.
For action epistemic model $(\Sigma, \sim, pre)$, pre defines a relation $\mathbf{S} \longrightarrow \mathbf{S} \bigotimes \mathbf{\Sigma}$ with $s \longrightarrow \{(s, \sigma): s \models {}_{\mathbf{s}}pre(\sigma)\}$ and this defines a function in the dual category C^{op} as below:

$$f : \mathbf{S} \bigotimes \mathbf{\Sigma} \longrightarrow \mathbf{S}$$
$$(s, \sigma) \longrightarrow s.$$

According to product update ($(s, \sigma) \models p$ iff $s \models p$) f is a measurable function. □

6.1 Epistemic measurable categories

We saw how finite epistemic models can define measurable spaces and how action epistemic models can define measurable functions. Measurable spaces that are produced by epistemic models and measurable functions that are produced by action epistemic models can define a category. It is sufficient to consider the objects of C^{op} without their relations (epistemic relations) and indeed they are measurable spaces. Also the arrows are single-agent action epistemic models with one member that we can consider those such as measurable functions. We denote this category by $\boldsymbol{C}^{op}$. Similarly we can define the category $\boldsymbol{C}^{op}{}_{\|.\|_S}$.

Theorem 6.4 *The category $\boldsymbol{C}^{op}{}_{\|.\|_S}$ is a subcategory of measurable spaces category*

Proof. With pervious theorems and definitions is obvious. □

7 Syntax, Semantics and Deduction Systems for Coalgebras Over Measurable Spaces

In this section we present our materials and notations from[10]. We will use syntax, semantics and deduction systems for coalgebra on measurable spaces for studying coalgebras over epistemic measurable categories.

Definition 7.1 [T-coalgebra] For a functor $T : Meas \longrightarrow Meas$ a T-coalgebra is a pair (X, α) where X is a measurable space and $\alpha : X \longrightarrow TX$ is a measurable function.

Definition 7.2 [T-coalgebra morphism] A T-coalgebra morphism$f$: $(X, \alpha) \longrightarrow (X^{'}, \alpha^{'})$ is given by a Meas-morphism $f : X \longrightarrow X^{'}$ that preserves the transition structures in the sense that $\alpha^{'} \circ f = Tf \circ \alpha$.

Definition 7.3 [Measurable polynomial functor] A measurable polynomial functor is any functor on Meas that can be constructed in finitely many steps from constant functors and/or the identity functor Id by forming products $T_1 \times T_2$, coproducts $T_1 + T_2$, exponentials T^E and measure-space functors $\triangle T$.

Definition 7.4 [The multigraph of ingredients] All the functors involved in the construction of T, along with the identity functor construct the ingredients of a measurable polynomial functor T . Define $IngT$ of ingredients inductively by putting:
- $IngT = \{T, Id\}$ if $T = Id$ or $T = X$.
- $IngT = \{T\} \cup IngT_1 \cup IngT_2$ if $T = T_{12}$ or $T = T_1 + T_2$.
- $IngT = \{T\} \cup IngS$ if $T = S^E$ or $T = \triangle S$.

We make $IngT$ a multigraph with labelled edges $\xrightarrow{k}$, $k \in \{pr_1, pr_2, in_1, in_2, eve, next, \geq p\}$. Also p is any rational number from $[0, 1]$ and e is an element of some set E occurring as an exponent in T.

We define the edges $\xrightarrow{k}$ joining ingredients of T as below:

- $S_1 \times S_2 \xrightarrow[T]{in_j} S_j$ and $S_1 + S_2 \xrightarrow[T]{pr_j} S_j$, for $j \in \{1, 2\}$;
- $S^E \xrightarrow[T]{ev_e} S$ for all $e \in S$;
- $\triangle S \xrightarrow[T]{P} S$ for $p \in [0, 1]_{\mathbb{Q}}$;
- $Id \xrightarrow[T]{next} T$.

7.1 Syntax and semantics

With ingredients of T we can define a many-sorted modal language for T-coalgebras, such as [12,18] and developed in [16] . For $S \xrightarrow{k} S^{'}$ in $IngT$, $[k]$ makes formulas of sort S out of formulas of sort $S^{'}$. $\varphi : S$ means that φ is a formula of sort S and $Form_S$ denote the set of all formulas of sort S for $\Gamma \subseteq Form_S$, $\Gamma : S$ means Γ is of sort S. $\varphi :: S$ means that $\varphi : S$ and every subformula of φ of constant sort is a measurable set. Also $\Gamma :: S$ means $\varphi :: S$ for all $\varphi \in \Gamma$.

Notation. Suppose that each constant ingredient of T is given by $\mathbb{X} = (X, \mathcal{A}_{\mathbb{X}}, \mathcal{A}_{\mathbb{X}}{}^{g})$ and $\mathcal{A}_{\mathbb{X}}{}^{g}$ being a fixed generating set for $\mathcal{A}_{\mathbb{X}}$.
For arbitrary ingredient S of T:
- $\perp_S : S$.
- $A : X$ if $A \in \mathcal{A}^{g}{}_{\mathbb{X}}$ or A is a singelton subset of X.
- If $\varphi_1 : S$ and $\varphi_2 : S$ then $\varphi_1 \longrightarrow \varphi_2 : S$.
- If $S \xrightarrow{k} S'$ in $IngT$ with $k \neq (\geq p)$ and $\varphi : S'$, then $[k]\varphi : S$.
- If $\triangle S \in IngT$ and $\varphi :: S$, then $[\geq p]\varphi : \triangle S$ for any $p \in [0,1]_Q$.

Each formula $\varphi : S$ in a T-coalgebra (X, α) can be interpreted as a subset $[[\varphi]]_S{}^{\alpha}$ of $S\mathbb{X}$, we can define it inductively as follows (Using $X \Rightarrow Y = (-X)$, $\beta^p(A) = \{\mu \mid \mu(A) \geq p\}$.):

$[[\perp_S]]_S{}^{\alpha} = \varnothing$;
$[[A]]_{\mathbb{X}'}{}^{\alpha} = A$;
$[[\varphi_1 \longrightarrow \varphi_2]]_S{}^{\alpha} = [[\varphi_1]]_S{}^{\alpha} \Rightarrow [[\varphi_2]]_S{}^{\alpha}$;
$[[\ [prj]\varphi\ ]]_{S_1 \times S_2}{}^{\alpha} = \pi^{-1}[[\varphi]]_{S_j}{}^{\alpha}$;
$[[\ [in_1]\varphi\ ]]_{S_1+S_2}{}^{\alpha} = in_1([[\varphi]]_{S_1}{}^{\alpha}) \cup in_2(S_2\mathbb{X})$;
$[[\ [in_2]\varphi\ ]]_{S_1+S_2}{}^{\alpha} = in_1(S_1\mathbb{X}) \cup in_2([[\varphi]]_{S_2}{}^{\alpha})$;
$[[\ [ev_e]\varphi\ ]]_{S^E}{}^{\alpha} = ev_e{}^{-1}[[\varphi]]_S{}^{\alpha}$;
$[[\ [next]\varphi\ ]]_{Id}{}^{\alpha} = \alpha^{-1}[[\varphi]]_T{}^{\alpha}$;
$[[\ [\geq P]\varphi\ ]]^{\alpha} = \beta^p[[\varphi]]_S{}^{\alpha}$.

Kripkean modal semantics can be introduced by $\alpha, x \models_S \varphi$ to mean that $x \in [[\varphi]]_S{}^{\alpha}$.

$$\begin{array}{lcl}
\alpha, x \not\models_S \perp_S; & & \\
\alpha, x \models_{\mathbb{X}} S & \Leftrightarrow & x \in A; \\
\alpha, x \models_S \varphi_1 \longrightarrow \varphi_2 & \Leftrightarrow & (\alpha, x \models_S \varphi_1 \quad \Rightarrow \quad \alpha, x \models_S \varphi_2); \\
\alpha, x \models_{S_1 \times S_2} [pr_j]\varphi & \Leftrightarrow & \alpha, \pi_j(x) \models_{S_j} \varphi; \\
\alpha, x \models_{S_1+S_2} [in_j]\varphi & \Leftrightarrow & (x = in_j(y) \quad \Rightarrow \quad \alpha, y \models_{S_j} \varphi); \\
\alpha, f \models_{S^E} [ev_e]\varphi & \Leftrightarrow & \alpha, f(e) \models_S \varphi; \\
\alpha, x \models_{Id} [next]\varphi & \Leftrightarrow & \alpha, \alpha(x) \models_T \varphi; \\
\alpha, \mu \models_{\triangle S} [\geq p]\varphi & \Leftrightarrow & \mu([[\varphi]]_S{}^{\alpha}) \geq p.
\end{array}$$

Also for modalities $[k]$ we can indroduce : $\alpha, x \models_S [k]\varphi$ iff $(xR_k y \quad \Rightarrow \quad \alpha, y \models_{S'} \varphi)$.

7.2 T-Deduction systems

Axioms: The set $Ax_S \subseteq Form_S$ of S-axioms is defined, for all $S \in IngT$, to consist of the following formulas.
1. All Boolean tautologies $\varphi : S$;
2. For $S = \mathbb{X}, A : \mathbb{X}$ and $c \in X$,
$(a)\ \{c\} \longrightarrow A \quad if\, c \in A$,
$(a)\ \{c\} \longrightarrow \neg A \quad if\, c \notin A$;

3. For $S = S_1 \times S_2, j \in \{1, 2\}$ and $\varphi : S_j$,
$(a) \neg[pr_j]\varphi \longrightarrow [pr_j]\neg\varphi$,
$(b) \neg[pr_j] \perp_{S_j}$;
4. For $S = S_1 + S_2$,
$(a) \neg[in_j]\varphi \longrightarrow [in_j]\neg\varphi$,
$(b) \neg[in_1] \perp_{S_1} \longleftrightarrow [in_2] \perp_{S_2}$;
5. For $S = U^E$ and $\varphi : U$,
$(a) \neg[ev_e]\varphi \longrightarrow [ev_e]\neg\varphi$,
$(b) \neg[ev_e] \perp_U$;
6. For $S = Id$ and $\varphi : T$,
$(a) \neg[next]\varphi \longrightarrow [next]\neg\varphi$,
$(b) \neg[next] \perp_T$;
7. For $S = \triangle S$,
$(a) [\geq 1](\varphi \longrightarrow \psi) \longrightarrow ([\geq p]\varphi \longrightarrow [\geq p]\psi)$,
$(b) [\geq p]\top_{S'}$,
$(c) [\geq p]\neg\varphi \longrightarrow \neg[\geq q]\varphi$ if $p + q > 1$,
$(d) [\geq p](\varphi \wedge \psi) \wedge [\geq q](\varphi \wedge \neg\psi) \longrightarrow [\geq p + q]\varphi$ if $p + q > 1$,
$(d) \neg[\geq p]\varphi \wedge \neg[\geq q]\psi \longrightarrow \neg[\geq p + q](\varphi \vee \psi)$ if $p + q > 1$.

Theorem 7.5 *For any $S \in IngT$, all S-axioms are valid in all T-coalgebras.*

Proof. See [10]. □

Definition 7.6 • $\Sigma \subseteq_w \Gamma$ means that Σ is a finite subset of Γ.
• $\bigwedge_w \Gamma$ is the set $\{\bigwedge \Sigma \mid \Sigma \subseteq_w \Gamma\}$ of conjunctions of all finite subsets of Γ.
• $\psi \longrightarrow \Gamma = \{\psi \longrightarrow \varphi \mid \varphi \in \Gamma\}$.
• For each edge $S \xrightarrow{k} S'$ and $\Gamma : S'$, define $[k]\Gamma = \{[k]\varphi \mid \varphi \in \Gamma\}$: S.

7.3 Deduction systems

Let $D = \{\vdash_s \mid S \in IngT\}$ be a collection of relations $\vdash_S \subseteq \mathcal{P}(Form_S) \times Form_S$. Then D is a T-deduction system if the following hold for all ingredients S:
• Assumption rule: $\varphi \in \Gamma \cup Ax_S$ implies $\Gamma \vdash_S \varphi$.
• Modus ponens: $\{\varphi, \varphi \longrightarrow \psi\} \vdash_S \psi$.
• Cut rule: If $\Gamma \vdash_S \psi$ for all $\psi \in \Sigma$ and $\Sigma \vdash_S \varphi$, then $\Gamma \vdash_S \varphi$.
• Deduction rule: $\Gamma \cup \{\varphi\} \vdash_S \psi$ implies $\Gamma \vdash_S \varphi \longrightarrow \psi$.
• Constant rule: If $\mathbb{X} \in IngT$, $\{\neg\{c\} \mid c \in X\} \vdash_{\mathbb{X}} \perp_{\mathbb{X}}$.
• Definite box rule: For each edge $S \xrightarrow{k} S'$ in $IngT$ with k a definite constructor, $\Gamma \vdash_{S'} \psi$ implies $[k]\Gamma \vdash_S [k]\psi$.
• Archimedean rule: If $\triangle S \in IngT$, $\{[\geq p]\varphi \mid p < q\} \vdash_{\triangle S} [\geq p]\varphi$.
• Countable additivity rule: If $\triangle S \in IngT$, then for countable $\Gamma :: S$, $\Gamma \vdash_S \psi$ implies $[\geq p](\bigwedge_w \Gamma) \vdash_{\triangle S} [\geq p]\psi$.

Also we can define local and global semantics consequence relations for $\Gamma \cup \{\varphi\} \subseteq Forms_S$ by:

$$\Gamma \models {_S}^{\alpha}\varphi \quad \Leftrightarrow \quad \forall x \in S\mathbb{X}, \alpha \quad (x \models {_S}\Gamma \Rightarrow \alpha, x \models {_S}\varphi);$$
$$\Gamma \models {_S}\varphi \quad \Leftrightarrow \quad \Gamma \models {_S}^{\alpha}\varphi \quad \forall T, \alpha.$$

Theorem 7.7 *(1) For any T-coalgebra $(\mathbb{X}, \alpha)$, the system $Conseq_T{}^{\alpha} = \{ \models {_S}^{\alpha} \mid S \in IngT \}$ of local consequence relations is a T-deduction system.*
(2) The global system $Conseq_T = \{ \models {_S} \mid S \in IngT \}$ is a T-deduction system.

Proof. See [10]. □

Interpretation (Formula) With our construction in a measurable space $\mathbb{X} = (X, \mathcal{A}_{\mathbb{X}}, \mathcal{A}_{\mathbb{X}}{}^g)$ we can interpret ontic proposition as measurable sets and doxastic proposition as subsets of X. (Epistemic relation) A measure on a measurable space can define different equivalence relations on it. For example with below definition of a equivalence relation we can consider a measurable space as an epistemic model with one cell information.

$$s \sim t \quad \text{iff} \quad \mu(\{s, t\}) \leq 1.$$

(Knowledge) We can interpret knowledge for a coalgebra $(\mathbb{X}, \alpha)$ and sort S as below:

$$K_S\varphi = [\geq p]\varphi \quad \alpha, \mu \models {_{\triangle S}}K\varphi \Longleftrightarrow \mu([[\varphi]]_S{}^{\alpha}) = 1.$$

Example 7.8 Suppose Ahmad hears from radio that one cities of Kerman is raining. Weatherman does not specify that which cities are now raining. Suppose that Kerman has five cities and Ahmad learns from four different ways that there isn't raining in four cities of Kerman. By omitting cities which there isn't raining in, Ahmad finds the raining city. Suppose that $\mathbf{S}_1$ is the initial epistemic model of Ahmad's knowledge which has five states s_1, s_2, s_3, s_4 and s_5 that are equivalent ($\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$). $\mathbf{S}_5$ is the final epistemic model of Ahmad's knowledge with one state s_5 that specifies the raining city.

$$\mathbf{S}_1 \xrightarrow{\sigma_{1,2}} \mathbf{S}_2 \xrightarrow{\sigma_{2,3}} \mathbf{S}_3 \xrightarrow{\sigma_{3,4}} \mathbf{S}_4 \xrightarrow{\sigma_{4,5}} \mathbf{S}_5 .$$

To study this protocol we define a category as below:
Objects $= \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5\}$.
Arrows$= \{\sigma_{1,1}, \sigma_{2,2}, \sigma_{3,3}, \sigma_{4,4}, \sigma_{5,5}, \sigma_{1,2}, \sigma_{1,3}, \sigma_{1,4}, \sigma_{1,5}, \sigma_{2,3}, \sigma_{2,4}, \sigma_{2,5}, \sigma_{3,4}, \sigma_{3,5}, \sigma_{4,5}\}$. With $Pre_{\sigma_{i,j}} = S_j$.
Above objects and arrows define an epistemic category. Total set T with $t_0 < t_1 < t_2 < t_3 < t_4$ members showing a time order for different actions that happened for Ahmad. We show this epistemic category by $C_{\|.\|_{\mathcal{S}}}$ and measurable epistemic category by $\boldsymbol{C}_{\|.\|_{\mathcal{S}}}$. Now we consider $\Box$ and $\Diamond$ on $\boldsymbol{C}^{op^T}{}_{\|.\|_{\mathcal{S}}}$.

$$\Box, \Diamond : {\boldsymbol{C}^{op}}^{T}{}_{\|\cdot\|_{\mathcal{S}}} \longrightarrow {\boldsymbol{C}^{op}}^{T}{}_{\|\cdot\|_{\mathcal{S}}}.$$

Indeed $\Box$ and $\Diamond$ are as bellow:

$$\Box = \mathbb{S}_1 \times \mathbb{S}_2 \times \mathbb{S}_3 \times \mathbb{S}_4 \times \mathbb{S}_5.$$
$$\Diamond = \mathbb{S}_1 + \mathbb{S}_2 + \mathbb{S}_3 + \mathbb{S}_4 + \mathbb{S}_5.$$

Also we have: $Ing\Box = Ing\Diamond = \{\mathbb{S}_1, \mathbb{S}_2, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5\}$. To study the above protocol we can study the coalgebras $(\mathbb{S}_i, \sigma_{i,j})$ with mentioned syntax, semantics and deduction systems.

8 Conclusion

We saw how we could present syntax, semantics and deduction systems for coalgebras over measurable spaces. Also we showed that $\boldsymbol{C}^{op}{}_{\|\cdot\|_{\mathcal{S}}}$ is a subcategory of measurable spaces. The question about capability of applying these syntax, semantics and deduction systems to $\boldsymbol{C}^{op}{}_{\|\cdot\|_{\mathcal{S}}}$ is natural. These conditions are closed under finite products, coprducts and Δ. We showed that $\boldsymbol{C}^{op}{}_{\|\cdot\|_{\mathcal{S}}}$ has finite products and coproducts. We don't consider any condition for Δ so $\boldsymbol{C}^{op}{}_{\|\cdot\|_{\mathcal{S}}}$ is closed under Δ. Also the functors $\Diamond$ and $\Box$ are polynomial. Therefore with $\boldsymbol{C}^{op}{}_{\|\cdot\|_{\mathcal{S}}}$ and temporal functors we can formalize epistemic protocols and present syntax, semantics and deduction systems for them. We can extend this settings for other modal changes because our settings are independent from modal relation. Indeed we can present a logical core for epistemic change and any other changes that can be applied to Kripke models and action Kripke models with the rule of update product.

9 Further Work

• It seems we can apply this setting for quantum changes. Baltag and Smets presented a formalization for quantum changes with dynamic logic [3,4].We can consider quantum models and action quantum models like of epistemic models and action epistemic models. Also linear temporal logic is compatible with quantum logic [9].
• Other formalizing of other intuition of time by functors is interesting. We can present several protocols with these functors for any epistemic category.
• We see the relation between amount of actions and information cells is important in the properties of epistemic categories. It seems we can have a discussion for this relation in a philosophical context.
• It's likely that we can consider our structure as a system core for different changes. Applying different changes by our materials and the developments of this system is still open.

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